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Determination of coefficients for a dissipative wave equation via boundary measurements

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Abstract

In this paper we consider the inverse problem of recovering the viscosity coefficient in a dissipative wave equation via boundary measurements. We obtain stability estimates by considering all possible measurements implemented on the boundary. We also prove that the viscosity coefficient is uniquely determined by a finite number of measurements on the boundary provided that it belongs to a given finite dimensional vector space.
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1. Introduction

Let us consider the following wave equation

$$\partial_{tt}u - \Delta u + q\partial_t u = 0 \quad \text{in } (0, T) \times \Omega, \tag{1.1}$$

where $T > 0$, Ω is a smooth bounded domain of $\mathbb{R}^N$, $N \geqslant 2$, and $q \in L^\infty(\Omega)$ is the viscosity coefficient.

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We are concerned with the inverse problem of recovering q in (1.1) via boundary measurements. More precisely, we are interested to recover q by giving boundary data f on $(0,T)\times\partial\Omega$ and measuring the corresponding flux $\partial u/\partial\nu$ on the same set.

These operations are described mathematically by the *Dirichlet-to-Neumann* operator Λ_q, which maps the trace of u into the corresponding flux. So, a general mathematical question concerning this inverse problem is to know if the knowledge of Λ_q uniquely determines q, i.e., if the map $q\mapsto\Lambda_q$ is invertible.

Taking into account the applications, it is important to precise this question. A first one is to know if the knowledge of $\Lambda_q(f)$ for all f determines q (*infinitely many measurements*). A second one is to know if the knowledge of $\Lambda_q(f_j)$, for $j=1,2,\dots,k$, determines q (*finite number of measurements*).

In this paper we consider these two questions. For the first one, we prove a stability estimate which implies that the map $q\mapsto\Lambda_q$ (defined on suitable spaces) has a continuous inverse. Concerning the second question, we prove that q can be uniquely determined by at most k boundary measurements $\Lambda_q(f_1),\Lambda_q(f_2),\dots,\Lambda_q(f_k)$ provided that q belongs to a known k-dimensional vector subspace of $L^\infty(\Omega)$.

Our main results are the following (with the norm $\|\Lambda_q\|$ defined in the sequel):

Theorem 1.1. *Assume that $q_1,q_2\in L^\infty(\Omega)$. If $T>\operatorname{diam}(\Omega)$, then there exists $C>0$ (depending only on N and Ω) such that*

$$\|q_1-q_2\|_{H^{-1/2}(\Omega)}\leqslant C\|\Lambda_{q_1}-\Lambda_{q_2}\|. \tag{1.2}$$

Moreover, if $q_1,q_2\in H^{\frac{N}{2}+s}(\Omega)$ for some $s>0$ and $\|q_j\|_{H^{\frac{N}{2}+s}}\leqslant M$, $j=1,2$, then, for each $0<r<s$, there exists $C_r>0$ (depending on M) such that

$$\|q_1-q_2\|_{H^{\frac{N}{2}+r}}\leqslant C_r\|\Lambda_{q_1}-\Lambda_{q_2}\|^{\theta(r)},$$

where $\theta(r)=2(s-r)/(N+2s+1)$. In particular, for each $0<r<s$ there exists $\tilde C_r$ such that

$$\|q_1-q_2\|_\infty\leqslant\tilde C_r\|\Lambda_{q_1}-\Lambda_{q_2}\|^{\theta(r)}.$$

Theorem 1.2. *Assume that $T>\operatorname{diam}(\Omega)$. Let $\{\rho_1,\rho_2,\dots,\rho_k\}$ a linearly independent subset of $L^\infty(\Omega)$ and consider*

$$\mathcal{X}:=\operatorname{span}\{\rho_1,\rho_2,\dots,\rho_k\}.$$

Then, for all $M>0$, there exist $f_1,f_2,\dots,f_k\in H^1(\Sigma)$ such that any $q\in\mathcal{X}$ satisfying $\|q\|_\infty\leqslant M$ is uniquely determined by $\Lambda_q(f_j)$, $j=1,\dots,k$.

The proof of Theorem 1.1 is presented in Section 2 and is based on the construction of high oscillatory geometric optics solutions introduced by Calderón [1] (see also [2]) and used by Rakesh and Symes [6] for the (conservative) wave equation. The proof of Theorem 1.2, which is inspired by the arguments used by Rakesh in [5], is presented in Section 3.

2. Stability estimates by infinite measurements on the boundary

Let $T > 0$, $\Omega \subset \mathbb{R}^N$, $N \geqslant 2$ a bounded domain with smooth boundary $\partial\Omega$ and $q \in L^\infty(\Omega)$. We denote $Q := (0,T) \times \Omega$ and $\Sigma := (0,T) \times \partial\Omega$. For $f \in H^1(\Sigma)$ we consider the initial-boundary value problem for the dissipative wave equation:

$$\begin{cases} \partial_t^2 u - \Delta u + q\partial_t u = 0, & (t,x) \in Q, \\ u(0,x) = \partial_t u(0,x) = 0, & x \in \Omega, \\ u(t,\sigma) = f(t,\sigma), & (t,\sigma) \in \Sigma. \end{cases} \tag{2.1}$$

It is well known (see [3, Theorem 2.1]) that if $f(0) = 0$, there exists a unique

$$u \in C\big([0,T]; H^1(\Omega)\big) \cap C^1\big([0,T]; L^2(\Omega)\big)$$

solution of (2.1) such that the Dirichlet-to-Neumann operator defined by

$$\Lambda_q : H^1(\Sigma) \to L^2(\Sigma), \qquad \Lambda_q(f) := \frac{\partial u}{\partial \nu},$$

is continuous. Therefore, we denote by $\|\Lambda_q\|$ its norm in $\mathcal{L}(H^1(\Sigma), L^2(\Sigma))$.

For $g \in H^1(\Sigma)$, we also consider the following backward-boundary value problem, called the *adjoint problem* of (2.1):

$$\begin{cases} \partial_t^2 v - \Delta v - q\partial_t v = 0, & (t,x) \in Q, \\ v(T,x) = \partial_t v(T,x) = 0, & x \in \Omega, \\ v(t,\sigma) = g(t,\sigma), & (t,\sigma) \in \Sigma \end{cases} \tag{2.2}$$

with the corresponding continuous Dirichlet-to-Neumann operator

$$\Lambda_q^* : H^1(\Sigma) \to L^2(\Sigma), \qquad \Lambda_q^*(g) := \frac{\partial v}{\partial \nu}.$$

Remark. It is easy to check that if $u(t,x)$ is the solution of (2.1) with boundary condition $f(t,\sigma)$, then $\tilde{u}$ defined by $\tilde{u}(t,x) := u(T-t,x)$ is the solution of (2.2) with boundary condition $\tilde{f}(t,\sigma) := f(T-t,\sigma)$. In particular, we have

$$\Lambda_q^*(\tilde{f}) = \widetilde{\Lambda_q(f)}.$$

The operators Λ_q and Λ_q^* satisfy the following properties:

Lemma 2.1. *We assume that $f, g \in H^1(\Sigma)$. Then we have*

$$\int_0^T \int_{\partial\Omega} \Lambda_q(f)(t,\sigma) g(t,\sigma)\, d\sigma\, dt = \int_0^T \int_{\partial\Omega} \Lambda_q^*(g)(t,\sigma) f(t,\sigma)\, d\sigma\, dt.$$

Proof. It is a direct consequence of integration by parts; we obtain the result once the equation in (2.1) is multiplied by $v(t,x)$ and integrated over Q. $\square$

Lemma 2.2. *Let $q_1, q_2 \in L^\infty(\Omega)$. We assume that u_1 is the solution of* (2.1) *for $q = q_1$ with boundary condition $f \in H^1(\Sigma)$ and v_2 is the solution of* (2.2) *for $q = q_2$ with boundary condition $g \in H^1(\Sigma)$. Then we have*

$$\int_0^T\int_\Omega \big(q_1(x) - q_2(x)\big)\partial_t u_1(t,x)v_2(t,x)\,dx\,dt = \int_0^T\int_{\partial\Omega} \big(\Lambda_{q_1}(f) - \Lambda_{q_2}(f)\big)g\,d\sigma\,dt.$$

Proof. It is a direct consequence of integration by parts and Lemma 2.1. □

Theorem 2.3. *Let $q \in L^\infty(\Omega)$, $\omega \in S^{N-1}$ and $\varphi, \psi \in C_0^\infty(\mathbb{R}^N)$ such that $\operatorname{supp}\varphi \cap \bar\Omega = (\operatorname{supp}\psi - T\omega) \cap \bar\Omega = \emptyset$. Then, there exists $C > 0$ depending on T, $\|q\|_{L^\infty(\Omega)}$, $\|\phi\|_{H^1(0,T;L^2(\Omega))}$ and $\|\psi\|_{H^2(0,T;L^2(\Omega))}$ such that, for all $\lambda > 0$, there exist*

$$R_{\lambda,q}, R^*_{\lambda,q} \in C^1\big([0,T]; L^2(\Omega)\big) \cap C\big([0,T]; H_0^1(\Omega)\big)$$

satisfying $R_{\lambda,q}(0) = \partial_t R_{\lambda,q}(0) = R^*_{\lambda,q}(T) = \partial_t R^*_{\lambda,q}(T) = 0$ a.e. in Ω and

$$\begin{cases} \|\partial_t R_{\lambda,q}\|_{L^2(0,T;L^2(\Omega))} \leqslant C, \\ \|\partial_t R_{\lambda,q}\|_{H^{-1}(0,T;L^2(\Omega))} + \|R_{\lambda,q}\|_{L^2(0,T;L^2(\Omega))} \leqslant \frac{C}{\lambda}, \\ \|R^*_{\lambda,q}\|_{L^2(0,T;L^2(\Omega))} \leqslant \frac{C}{\lambda}, \end{cases} \tag{2.3}$$

for which the functions $u(t,x)$ and $v(t,x)$ defined by

$$\begin{cases} u(t,x) := \frac{1}{\mathrm{i}\lambda}\varphi(x+t\omega)\exp(\mathrm{i}\lambda(x\cdot\omega + t)) + R_{\lambda,q}(t,x), \\ v(t,x) := \psi(x+t\omega)\exp(-\mathrm{i}\lambda(x\cdot\omega + t)) + R^*_{\lambda,q}(t,x), \end{cases} \tag{2.4}$$

are respectively solutions of (2.1) *and* (2.2). *Moreover, the map $q \mapsto \partial_t R_{\lambda,q}$ is locally Lipschitz continuous from $L^\infty(\Omega)$ into $L^2(0,T;L^2(\Omega))$. More precisely, if $\|q_j\|_\infty \leqslant M$, $j = 1,2$, there exists a constant $C = C(M) > 0$ such that*

$$\begin{cases} \|\partial_t R_{\lambda,q_1} - \partial_t R_{\lambda,q_2}\|_{L^2(0,T;L^2(\Omega))} \leqslant C\|q_1 - q_2\|_\infty, \\ \|\partial_t R_{\lambda,q_1} - \partial_t R_{\lambda,q_2}\|_{H^{-1}(0,T;L^2(\Omega))} \leqslant \frac{C}{\lambda}\|q_1 - q_2\|_\infty. \end{cases} \tag{2.5}$$

The proof of Theorem 2.3 is based on the following lemmas.

Lemma 2.4. *Let $\lambda > 0$, $\omega \in S^{N-1}$, $q \in L^\infty(\Omega)$ and $\phi_1 \in L^2(0,T;L^2(\Omega))$. Let R be the unique solution of*

$$\begin{cases} \partial_t^2 R - \Delta R + q\partial_t R = \frac{1}{\lambda}h_1, \\ R(0) = \partial_t R(0) = 0, & x \in \Omega, \\ R(t,\sigma) = 0, & (t,\sigma) \in \Sigma, \end{cases} \tag{2.6}$$

where $h_1(t,x) := \phi_1(t,x)\exp(\mathrm{i}\lambda(x\cdot\omega + t))$. Then, there exists $C > 0$ (depending on T, $\|\phi_1\|_{L^2(0,T;L^2(\Omega))}$ and $\|q\|_\infty$, but independent of λ) such that

$$\|\partial_t R\|_{L^2(0,T;L^2(\Omega))} + \|R\|_{L^2(0,T;L^2(\Omega))} \leqslant \frac{C}{\lambda}.$$

Proof. Multiplying both sides of the equation in (2.6) by $\overline{\partial_t R}$, integrating over Ω and taking its real part, we get

$$\frac{d}{dt}\left(\int_\Omega \left|\partial_t R(t)\right|^2 dx + \int_\Omega \left|\nabla R(t)\right|^2 dx\right) \leqslant \frac{1}{\lambda^2}\int_\Omega \left|h_1(t)\right|^2 dx + \left(1+2\|q\|_\infty\right)\int_\Omega |\partial_t R|^2\, dx.$$

Thanks to the Gronwall inequality, we obtain

$$\left\|\partial_t R(t)\right\|_2^2 + \left\|\nabla R(t)\right\|_2^2 \leqslant \frac{C_1}{\lambda^2},$$

where $C_1 > 0$ is a constant which depends on T, $\|\phi_1\|_{L^2(0,T;L^2(\Omega))}$ and $\|q\|_\infty$. Therefore, since $R \in C([0,T]; H_0^1(\Omega))$, the conclusion follows from the Poincaré inequality. □

Lemma 2.5. *Let $\lambda > 0$, $\omega \in S^{N-1}$, $q \in L^\infty(\Omega)$ and $\phi_2 \in H^1(0,T; L^2(\Omega))$ such that $\phi_2(0,x) = 0$. Let R be the unique solution of*

$$\begin{cases} \partial_t^2 R - \Delta R + q\,\partial_t R = h_2, & \\ R(0) = \partial_t R(0) = 0, & x \in \Omega, \\ R(t,\sigma) = 0, & (t,\sigma) \in \Sigma, \end{cases} \tag{2.7}$$

where $h_2(t,x) := \phi_2(t,x)\exp(\mathrm{i}\lambda(x\cdot\omega + t))$. Then, there exists a constant $C > 0$ (depending on T, $\|\phi_2\|_{H^1(0,T;L^2(\Omega))}$ and $\|q\|_\infty$, but independent of λ) such that

$$\begin{cases} \|\partial_t R\|_{L^2(0,T;L^2(\Omega))} \leqslant C, \\ \|\partial_t R\|_{H^{-1}(0,T;L^2(\Omega))} + \|R\|_{L^2(0,T;L^2(\Omega))} \leqslant \frac{C}{\lambda}. \end{cases}$$

Proof. The first inequality is obtained using the same arguments as in the proof of Lemma 2.4.

We consider $w(t,x) := \int_0^t R(\tau,x)\,d\tau$ and $h(t,x) := \int_0^t h_2(\tau,x)\,d\tau$, where R is the solution of (2.7). Then,

$$\begin{cases} \partial_t^2 w - \Delta w + q\,\partial_t w = h; & \\ w(0) = \partial_t w(0) = 0 & x \in \Omega, \\ w(t,\sigma) = 0 & (t,\sigma) \in \Sigma. \end{cases}$$

Since we have

$$h(t,x) = \frac{1}{\mathrm{i}\lambda}\left(\phi_2(t,x)\exp\big(\mathrm{i}\lambda(x\cdot\omega + t)\big) - \int_0^t \partial_t\phi_2(\tau,x)\exp\big(\mathrm{i}\lambda(x\cdot\omega+\tau)\big)\,d\tau\right),$$

the same arguments in the proof of Lemma 2.4 give

$$\left\|\partial_t w(t)\right\|_2^2 + \left\|\nabla w(t)\right\|_2^2 \leqslant \frac{C_1}{\lambda^2},$$

where $C_1 > 0$ depends on $\|\phi_2\|_{H^1(0,T;L^2(\Omega))}$ and $\|q\|_\infty$. In particular,

$$\|R\|_{L^2(0,T;L^2(\Omega))} = \|\partial_t w\|_{L^2(0,T;L^2(\Omega))} \leqslant \frac{C_1}{\lambda}. \tag{2.8}$$

Since $\partial_t : L^2(0,T;L^2(\Omega)) \to H^{-1}(0,T;L^2(\Omega))$ is a continuous operator, there exists a constant $C_2 > 0$ such that

$$\|\partial_t R\|_{H^{-1}(0,T;L^2(\Omega))} \leqslant C_2 \|R\|_{L^2(0,T;L^2(\Omega))} \tag{2.9}$$

and the conclusion follows from (2.8) and (2.9). □

Corollary 2.6. *Let $\lambda > 0$, $\omega \in S^{N-1}$ and $\psi_1 \in H^1(0,T;L^2(\Omega))$ such that $\psi_1(T,x) = 0$. Let R be the solution of*

$$\begin{cases} \partial_t^2 R - \Delta R - q\partial_t R = h_1^*, & \\ R(T) = \partial_t R(T) = 0, & x \in \Omega, \\ R(t,\sigma) = 0, & (t,\sigma) \in \Sigma, \end{cases}$$

where $h_1^*(t,x) := \psi_1(t,x)\exp(-\mathrm{i}\lambda(x\cdot\omega + t))$. Then, there exists a constant $C > 0$ (depending on T, $\|q\|_\infty$ and $\|\psi_1\|_{H^1(0,T;L^2(\Omega))}$, but independent of λ) such that

$$\|\partial_t R\|_{H^{-1}(0,T;L^2(\Omega))} + \|R\|_{L^2(0,T;L^2(\Omega))} \leqslant \frac{C}{\lambda}.$$

Lemma 2.7. *Let $\lambda > 0$, $\omega \in S^{N-1}$ and $\psi_2 \in H^2(0,T;L^2(\Omega))$ such that $\psi_2(T,x) = \partial_t\psi_2(T,x) = 0$. Let R be the solution of*

$$\begin{cases} \partial_t^2 R - \Delta R - q\partial_t R = \lambda h_2^*, & \\ R(T) = \partial_t R(T) = 0, & x \in \Omega, \\ R(t,\sigma) = 0, & (t,\sigma) \in \Sigma, \end{cases} \tag{2.10}$$

where $h_2^*(t,x) := \psi_2(t,x)\exp(-\mathrm{i}\lambda(x\cdot\omega + t))$. Then, there exists a constant $C > 0$ (depending on T, $\|\psi_2\|_{H^2(0,T;L^2(\Omega))}$ and $\|q\|_\infty$, but independent of λ) such that

$$\|R\|_{H^{-1}(0,T;L^2(\Omega))} \leqslant \frac{C}{\lambda}. \tag{2.11}$$

Proof. We consider

$$\zeta(t,x) := \int_t^T\int_\tau^T R(s,x)\,ds\,d\tau, \qquad \vartheta(t,x) := \int_t^T\int_\tau^T \lambda h_2^*(s,x)\,ds\,d\tau. \tag{2.12}$$

Then it is easy to see that

$$\begin{cases} \partial_t^2 \zeta - \Delta\zeta - q\partial_t\zeta = \vartheta; & \\ \zeta(0) = \partial_t\zeta(0) = 0 & \text{in } \Omega, \\ \zeta(t,\sigma) = 0, & (t,\sigma) \in \Sigma. \end{cases} \tag{2.13}$$

After integrating by parts two times the second integral in (2.12), we obtain

$$\|\vartheta\|_{L^2(0,T;L^2(\Omega))} \leqslant \frac{C}{\lambda}.$$

Multiplying the equation in (2.13) by $\overline{\partial_t\zeta}$, integrating over Ω and taking its real part, we obtain

$$\|\partial_t \zeta\|_{L^2(0,T;L^2(\Omega))} \leqslant \frac{C}{\lambda}$$

and the conclusion follows from the continuity of ∂_t. □

Proof of Theorem 2.3. Let $\omega \in S^{N-1}$ and consider u the function defined by

$$u(t,x) := \frac{1}{\mathrm{i}\lambda}\varphi(x+t\omega)e^{\mathrm{i}\lambda(x\cdot\omega+t)} + R(t,x).$$

Then, by a direct calculation we have

$$\begin{aligned}\left(\partial_t^2 - \Delta + q\partial_t\right)u = {}& \frac{1}{\mathrm{i}\lambda}\left[\sum_{i,j=1}^{N}\omega_i\omega_j\frac{\partial^2\varphi}{\partial x_i\partial x_j} - \Delta\varphi + q\omega\cdot\nabla\varphi\right]e^{\mathrm{i}\lambda(x\cdot\omega+t)} \\ & + q\varphi e^{\mathrm{i}\lambda(x\cdot\omega+t)} + \left(\partial_t^2 - \Delta + q\partial_t\right)R.\end{aligned}$$

Now consider

$$\phi_1(t,x) := \mathrm{i}\left[\sum_{i,j=1}^{N}\omega_i\omega_j\frac{\partial^2\varphi}{\partial x_i\partial x_j}(x+t\omega) - \Delta\varphi(x+t\omega) + q(x)\omega\cdot\nabla\varphi(x+t\omega)\right],$$
$$\phi_2(t,x) := -q(x)\varphi(x+t\omega).$$

Since we are assuming that $\operatorname{supp}\varphi \cap \bar{\Omega} = \emptyset$, it follows that $\phi_j \in H^1(0,T;L^2(\Omega))$, $\phi_j(0,x) = 0$, $j = 1,2$. Therefore, if $R_{\lambda,q}$ is the solution of

$$\begin{cases}\partial_t^2 R - \Delta R + q\partial_t R = \frac{1}{\lambda}h_1 + h_2, & \\ R(0) = \partial_t R(0) = 0, & x \in \Omega, \\ R(t,\sigma) = 0, & (t,\sigma) \in \Sigma,\end{cases} \tag{2.14}$$

where $h_j(t,x) := \phi_j(t,x)\exp(\mathrm{i}\lambda(x\cdot\omega+t))$, then we have that u is a solution of (2.1) and the inequalities in (2.3) follows from Lemmas 2.4 and 2.5.

By the same arguments,

$$\begin{aligned}\left(\partial_t^2 - \Delta - q\partial_t\right)v = {}& \left[\sum_{i,j=1}^{N}\omega_i\omega_j\frac{\partial^2\psi}{\partial x_i\partial x_j} - \Delta\psi - q\omega\cdot\nabla\psi\right]e^{-\mathrm{i}\lambda(x\cdot\omega+t)} \\ & + \mathrm{i}\lambda q\psi e^{-\mathrm{i}\lambda(x\cdot\omega+t)} + \left(\partial_t^2 - \Delta - q\partial_t\right)R^*.\end{aligned}$$

If we denote

$$\psi_1(t,x) := -\left[\sum_{i,j=1}^{N}\omega_i\omega_j\frac{\partial^2\psi}{\partial x_i\partial x_j}(x+t\omega) - \Delta\psi(x+t\omega) - q(x)\omega\cdot\nabla\psi(x+t\omega)\right],$$
$$\psi_2(t,x) := -\mathrm{i}q(x)\psi(x+t\omega),$$

then, $\psi_j \in H^2(0,T;L^2(\Omega))$ and $\psi_j(T,x) = \partial_t\psi_j(T,x) = 0$, $j = 1,2$ because $(\operatorname{supp}\psi - T\omega) \cap \bar{\Omega} = \emptyset$.

Therefore, if $R^*_{\lambda,q}$ is the solution of

$$\begin{cases}\partial_t^2 R - \Delta R - q\partial_t R = h_1^* + \lambda h_2^*, & \\ R(T) = \partial_t R(T) = 0, & x \in \Omega, \\ R(t,\sigma) = 0, & (t,\sigma) \in \Sigma,\end{cases} \tag{2.15}$$

where $h_j^*(t,x) := \psi_j(t,x)e^{-\mathrm{i}\lambda(x\cdot\omega+t)}$, then v is solution of (2.2) and the second inequality in (2.3) follows from Corollary 2.6 and Lemma 2.7.

We are now in position to prove inequalities (2.5). Let $q_1, q_2 \in L^\infty(\Omega)$ such that $\|q_j\|_\infty \leqslant M$. Define $S := R_{\lambda,q_1} - R_{\lambda,q_2}$. Then S is the solution of

$$\begin{cases} \partial_t^2 S - \Delta S + q_1 \partial_t S = (q_2 - q_1)\partial_t R_{\lambda,q_2} + \left(\frac{1}{\lambda}\phi_1 + \phi_2\right)(q_1 - q_2)e^{\mathrm{i}\lambda(t+\omega\cdot x)}, \\ S(0) = \partial_t S(0) = 0, \quad x \in \Omega, \\ S(t,\sigma) = 0, \quad (t,\sigma) \in \Sigma, \end{cases} \tag{2.16}$$

where $\phi_1(t,x) := \mathrm{i}\omega \cdot \nabla\varphi(x + t\omega)$ and $\phi_2(t,x) := -\varphi(x + t\omega)$.

We decompose S as $S = S_1 + S_2 + S_3$, where

$$\begin{cases} \partial_t^2 S_1 - \Delta S_1 + q_1 \partial_t S_1 = (q_2 - q_1)\partial_t R_{\lambda,q_2}, \\ \partial_t^2 S_2 - \Delta S_2 + q_1 \partial_t S_2 = \frac{1}{\lambda}(q_1 - q_2)\phi_1 e^{\mathrm{i}\lambda(t+\omega\cdot x)}, \\ \partial_t^2 S_3 - \Delta S_3 + q_1 \partial_t S_3 = (q_1 - q_2)\phi_2 e^{\mathrm{i}\lambda(t+\omega\cdot x)}. \end{cases}$$

With the same arguments used in the proof of Lemmas 2.4 and 2.5, and the first inequality in (2.3), we obtain that $\|\partial_t S_j\|_{H^{-1}(0,T,L^2(\Omega))} \leqslant \frac{C}{\lambda}\|q_1 - q_2\|_\infty$, $j = 2, 3$ and $\|\partial_t S_j\|_{L^2(0,T,L^2(\Omega))} \leqslant C\|q_1 - q_2\|_\infty$, $j = 1, 2, 3$.

In order to prove that the same estimate holds for S_1, let $w(t,x)$ be the function defined by $w(t,x) := \int_0^t S_1(\tau,x)\,d\tau$. Since $R_{\lambda,q_2}(0,x) = 0$, it is easy to see that w satisfies

$$\begin{cases} \partial_t^2 w - \Delta w + q_1 \partial_t w = (q_2 - q_1)R_{\lambda,q_2}, \\ w(0) = \partial_t w(0) = 0, \quad x \in \Omega, \\ w(t,\sigma) = 0, \quad (t,\sigma) \in \Sigma. \end{cases}$$

Multiplying the equation above by $\overline{\partial_t w}$, integrating over Ω and taking the real part, we have

$$\frac{1}{2}\frac{d}{dt}\left(\left\|\partial_t w(t)\right\|_2^2 + \left\|\nabla w(t)\right\|_2^2\right) + \int_\Omega q_1 \left|\partial_t w(s)\right|^2 dx$$
$$\leqslant \|q_1 - q_2\|_\infty \left\|R_{\lambda,q_2}(t)\right\|_2 \left\|\partial_t w(t)\right\|_2.$$

Therefore,

$$\left\|\partial_t w(t)\right\|_2^2 + \left\|\nabla w(t)\right\|_2^2 \leqslant \|q_1 - q_2\|_\infty^2 \|R_{\lambda,q_2}\|_{L^2(0,T;L^2(\Omega))}^2 + (1 + 2M)\int_0^t \left\|\partial_t w(s)\right\|_2^2 ds.$$

It follows from the Gronwall inequality and (2.3) that

$$\left\|S_1(t)\right\|_2^2 = \left\|\partial_t w(t)\right\|_2^2 \leqslant \frac{C}{\lambda^2}\|q_1 - q_2\|_\infty^2$$

and hence

$$\|S_1\|_{L^2(0,T;L^2(\Omega))} \leqslant \frac{C}{\lambda}\|q_1 - q_2\|_\infty.$$

By the continuity of the operator ∂_t, we have

$$\|\partial_t S_1\|_{H^{-1}(0,T;L^2(\Omega))} \leqslant C\|S_1\|_{L^2(0,T;L^2(\Omega))}$$

and the proof is complete. □

Proof of Theorem 1.1. Let $\varphi, \psi \in C_0^\infty(\Omega_\varepsilon)$, where

$$\varepsilon = \frac{T - \operatorname{diam}\Omega}{2} \quad \text{and} \quad \Omega_\varepsilon = \left\{x \in \mathbb{R}^N \setminus \bar{\Omega};\ \operatorname{dist}(x, \Omega) < \varepsilon\right\}.$$

We have $\operatorname{supp}\varphi \cap \bar{\Omega} = \emptyset$, $(\operatorname{supp}\psi - T\omega) \cap \bar{\Omega} = \emptyset$ and it follows from Theorem 2.3 that we can consider u_1 and v_2 respectively solutions of (2.1) and (2.2) satisfying (2.4).

By Lemma 2.2, it follows that

$$\left|\int_0^T\int_\Omega (q_1 - q_2)\partial_t u_1 v_2\,dx\,dt\right| \leqslant \left\|\Lambda_{q_1}(f_\lambda) - \Lambda_{q_2}(f_\lambda)\right\|_{L^2(\Sigma)}\|g_\lambda\|_{L^2(\Sigma)}$$
$$\leqslant \|\Lambda_{q_1} - \Lambda_{q_2}\|\,\|f_\lambda\|_{H^1(\Sigma)}\|g_\lambda\|_{L^2(\Sigma)}, \tag{2.17}$$

where

$$\begin{cases} f_\lambda(t,\sigma) := \frac{1}{\mathrm{i}\lambda}\varphi(\sigma + t\omega)\exp(\mathrm{i}\lambda(t + \omega\cdot\sigma)), \\ g_\lambda(t,\sigma) := \psi(\sigma + t\omega)\exp(-\mathrm{i}\lambda(t + \omega\cdot\sigma)), \end{cases} \quad (t,\sigma) \in \Sigma. \tag{2.18}$$

Considering the special form of u_1 and v_2, we have from (2.17):

$$\left|\int_0^T\int_\Omega \big(q_1(x) - q_2(x)\big)\varphi(x + t\omega)\psi(x + t\omega)\,dx\,dt\right| \leqslant \alpha_\lambda + \beta_\lambda + \gamma_\lambda + \delta_\lambda, \tag{2.19}$$

where

$$\begin{cases} \alpha_\lambda := \frac{1}{\lambda}\|q_1 - q_2\|_\infty \int_0^T\int_\Omega |(\omega\cdot\nabla\varphi)(|\psi| + |R^*_{\lambda,q_2}|)|\,dx\,dt, \\ \beta_\lambda := \|q_1 - q_2\|_\infty \int_0^T\int_\Omega |\varphi R^*_{\lambda,q_2}|\,dx\,dt, \\ \gamma_\lambda := \left|\int_0^T\int_\Omega (q_1 - q_2)(\psi\exp(-\mathrm{i}\lambda(x\cdot\omega + t)) + R^*_{\lambda,q_2})\partial_t R_{\lambda,q_1}\,dx\,dt\right|, \\ \delta_\lambda := \|\Lambda_{q_1} - \Lambda_{q_2}\|\,\|f_\lambda\|_{H^1(\Sigma)}\|g_\lambda\|_{L^2(\Sigma)}. \end{cases} \tag{2.20}$$

Since $\varphi \in C_0^\infty(\Omega_\varepsilon)$, it follows from the choice of ε that the maps

$$(t,x) \mapsto \varphi(x + t\omega), \qquad (t,x) \mapsto \psi(x + t\omega) \quad \text{and} \quad (t,x) \mapsto \omega\cdot\nabla\varphi(x + t\omega)$$

belong to $H_0^1(0,T;L^2(\Omega))$. Hence, we have from Theorem 2.3 that

$$\alpha_\lambda \leqslant C/\lambda, \qquad \beta_\lambda \leqslant C/\lambda \quad \text{and} \quad \gamma_\lambda \leqslant C/\lambda.$$

It is clear from (2.18) that there exists a constant $C > 0$ (depending on Ω but independent of λ) such that $\|g_\lambda\|_{L^2(\Sigma)} \leqslant C\|\psi\|_{L^2(\mathbb{R}^N)}$. Moreover, it is not difficult to show that there exists another constant $C > 0$ such that

$$\|f_\lambda\|^2_{H^1(\Sigma)} \leqslant C\left(\frac{1}{\lambda}\|\varphi\|_{H^1(\mathbb{R}^N)} + \|\varphi\|^2_{L^2(\mathbb{R}^N)}\right),$$

and we obtain, after taking the limit as $\lambda \to \infty$ on the right side of (2.19),

$$\left|\int_0^T\int_\Omega \big(q_1(x)-q_2(x)\big)\varphi(x+t\omega)\psi(x+t\omega)\,dx\,dt\right|$$
$$\leqslant C\|\Lambda_{q_1}-\Lambda_{q_2}\|\,\|\varphi\|_{L^2(\mathbb{R}^N)}\|\psi\|_{L^2(\mathbb{R}^N)}.$$

Let us denote by $\rho(x)$ the zero extension of $q_1(x)-q_2(x)$ on $\mathbb{R}^N\setminus\Omega$. Then, if follows from Fubini theorem that

$$\left|\int_{\mathbb{R}^N}\left[\int_0^T \rho(y-s\omega)\,ds\right]\varphi(y)\psi(y)\,dy\right| \leqslant C\|\Lambda_{q_1}-\Lambda_{q_2}\|\,\|\varphi\|_{L^2(\mathbb{R}^N)}\|\psi\|_{L^2(\mathbb{R}^N)}. \tag{2.21}$$

By density, we obtain

$$\left|\int_{\mathbb{R}^N}\left[\int_0^T \rho(y-s\omega)\,ds\right]\Phi(y)\,dy\right| \leqslant C\|\Lambda_{q_1}-\Lambda_{q_2}\|,$$

where Φ is an arbitrary function of $L^1(\Omega_\varepsilon)$ satisfying $\|\Phi\|_{L^1(\Omega_\varepsilon)}\leqslant 1$. Hence, we have by density,

$$\left|\int_0^T \rho(y-s\omega)\,ds\right| \leqslant C\|\Lambda_{q_1}-\Lambda_{q_2}\|, \quad \text{a.e. in } \Omega_\varepsilon.$$

Since $T>\operatorname{diam}(\Omega)$, $\omega\in S^{N-1}$ and $\operatorname{supp}\rho\subset\Omega$, we obtain

$$\big|P[\rho](\omega,y)\big| = \left|\int_{-\infty}^{\infty} \rho(y-s\omega)\,ds\right| \leqslant 2C\|\Lambda_{q_1}-\Lambda_{q_2}\|, \quad \text{a.e. in } \mathbb{R}^N. \tag{2.22}$$

For $R>0$ such that $\Omega\subset B_R$, we obtain

$$\big\|P[\rho]\big\|^2_{L^2(\mathcal{T})} := \int_{S^{N-1}}\int_{\omega^\perp\cap B_R} \big|P[\rho](\omega,y)\big|^2\,dy\,d\omega \leqslant C\|\Lambda_{q_1}-\Lambda_{q_2}\|^2, \tag{2.23}$$

where $\mathcal{T}:=\{(\omega,y);\ \omega\in S^{N-1},\ y\in\omega^\perp\}$ is the tangent bundle.

For the X-ray transform, we have the following well-known estimate (see [4]):

$$\|\rho\|_{H^{-1/2}(\Omega)} \leqslant C\big\|P[\rho]\big\|_{L^2(\mathcal{T})}, \tag{2.24}$$

where $C>0$ depends only on N. Combining (2.23) and (2.24), we obtain (1.2). The conclusion follows from interpolation formulæ and classical Sobolev imbedding theorems. $\square$

3. Recovery by finite number of boundary measurements

In this section we assume that $\{\rho_1, \rho_2, \dots, \rho_k\}$ is a given linearly independent set of functions of $L^\infty(\mathbb{R}^N)$ such that $\operatorname{supp} \rho_i \subset \Omega$ and $\mathcal{X} := \operatorname{span}\{\rho_1, \rho_2, \dots, \rho_k\}$. For each $\omega \in S^{N-1}$ we denote by $P_\omega[\rho_i]$ the X-ray transform of ρ_i in the direction ω, i.e.,

$$P_\omega[\rho_i](x) := \int_{-\infty}^{\infty} \rho_i(x + t\omega)\, dt$$

and, for each $d > 0$, we denote $\Omega_d := \{x \in \mathbb{R}^N \setminus \bar{\Omega};\ \operatorname{dist}(x, \Omega) < d\}$.

Lemma 3.1. *For all $d > 0$, there exist $\omega_j \in S^{N-1}$ and $\varphi_j \in C_0^\infty(\Omega_d)$, $j = 1, \dots, k$, such that the matrix $\mathcal{A} = (a_{ij})$, with entries defined by*

$$a_{ij} := \int_{\mathbb{R}^N} P_{\omega_j}[\rho_i](x)\varphi_j^2(x)\, dx, \tag{3.1}$$

is invertible.

Proof. We proceed by induction on k. For each $\omega \in S^{N-1}$ and $\varphi \in C_0^\infty(\Omega_d)$, we define

$$a_i(\omega, \varphi) := \int_{\mathbb{R}^N} P_\omega[\rho_i](x)\varphi^2(x)\, dx.$$

If $k = 1$, there exist ω_1 and φ_1 such that $a_1(\omega_1, \varphi_1) \neq 0$, because, otherwise we would have $P_\omega[\rho_1](x) = 0$ a.e. in Ω_d, for all $\omega \in S^{N-1}$ and, as a consequence of the properties of the X-ray transform, $\rho_1 \equiv 0$.

We assume now that the result is true for any linearly independent set of $k - 1$ functions and we consider $\mathcal{X} := \operatorname{span}\{\rho_1, \rho_2, \dots, \rho_k\}$.

If the result is not true for $\mathcal{X}$, we have

$$\det \begin{pmatrix} a_1(\omega_1, \varphi_1) & \cdots & a_1(\omega_k, \varphi_k) \\ \vdots & \ddots & \vdots \\ a_k(\omega_1, \varphi_1) & \cdots & a_k(\omega_k, \varphi_k) \end{pmatrix} = 0,$$

for all $\omega_1, \dots, \omega_k \in S^{N-1}$ and $\varphi_1, \dots, \varphi_k \in C_0^\infty(\Omega_d)$. In particular, for fixed ω_j and φ_j, $j = 2, \dots, k$, we have

$$C_1 a_1(\omega_1, \varphi_1) + C_2 a_2(\omega_1, \varphi_1) + \cdots + C_k a_k(\omega_1, \varphi_k) = 0,$$
$$\forall \omega_1 \in S^{N-1},\ \forall \varphi_1 \in C_0^\infty(\Omega_d),$$

where C_j is the cofactor of $a_j(\omega_1, \varphi_1)$.

Using the induction hypothesis, we can chose $\omega_2, \dots, \omega_k$ and $\varphi_2, \dots, \varphi_k$ such that C_j are not all zero. It follows that

$$\int_{\mathbb{R}^N} P_{\omega_1}[C_1\rho_1 + \cdots + C_k\rho_k](x)\varphi_1^2(x)\, dx = 0, \quad \forall \omega_1 \in S^{N-1},\ \forall \varphi_1 \in C_0^\infty(\Omega_d).$$

Hence, $P_{\omega_1}[C_1\rho_1+\cdots+C_k\rho_k](x)=0$ a.e. in Ω_d, for all $\omega_1\in S^{N-1}$ and, consequently, $C_1\rho_1+\cdots+C_k\rho_k\equiv 0$, which is a contradiction.

Hence the proof is complete. □

We are now in position to prove our second main result.

Proof of Theorem 1.2. For $\varepsilon := (T-\operatorname{diam}(\Omega))/2$, we consider

$$\Omega_\varepsilon=\bigl\{x\in\mathbb{R}^N\setminus\bar\Omega;\ \operatorname{dist}(x,\Omega)<\varepsilon\bigr\}.$$

Let $\varphi\in C_0^\infty(\Omega_\varepsilon)$. Then $\operatorname{supp}\varphi\cap\bar\Omega=\emptyset$ and $(\operatorname{supp}\varphi-T\omega)\cap\bar\Omega=\emptyset$. Hence, we can consider the functions u and v defined by (2.4) (with $\psi=\varphi$), where u is a solution of (2.1) with $q_1\in\mathcal{X}$ and v is a solution of (2.2) with $q_2\in\mathcal{X}$, where the parameter $\lambda>0$ will be chosen a posteriori.

It follows from Lemma 2.2 that

$$\int_0^T\int_\Omega(q_1-q_2)\partial_t u v\,dx\,dt=\int_0^T\int_{\partial\Omega}\bigl(\Lambda_{q_1}(f)-\Lambda_{q_2}(f)\bigr)g\,d\sigma\,dt. \tag{3.2}$$

Considering the special form of u_1 and v_2, we have from above

$$\left|\int_0^T\int_\Omega\bigl(q_1(x)-q_2(x)\bigr)\bigl(\varphi(x+t\omega)\bigr)^2\,dx\,dt\right|\leqslant\alpha_\lambda+\beta_\lambda+\gamma_\lambda+\delta_\lambda, \tag{3.3}$$

where

$$\begin{cases}\alpha_\lambda:=\frac{1}{\lambda}\|q_1-q_2\|_\infty\int_0^T\int_\Omega|(\omega\cdot\nabla\varphi)(|\varphi|+|R^*_{\lambda,q_2}|)|\,dx\,dt,\\ \beta_\lambda:=\|q_1-q_2\|_\infty\int_0^T\int_\Omega|\varphi R^*_{\lambda,q_2}|\,dx\,dt,\\ \gamma_\lambda:=\bigl|\int_0^T\int_\Omega(q_1-q_2)(\varphi\exp(-\mathrm{i}\lambda(x\cdot\omega+t))+R^*_{\lambda,q_2})\partial_t R_{\lambda,q_1}\,dx\,dt\bigr|,\\ \delta_\lambda:=\|\Lambda_{q_1}(f_\lambda)-\Lambda_{q_2}(f_\lambda)\|_{L^2(\Sigma)}\|g_\lambda\|_{L^2(\Sigma)}.\end{cases} \tag{3.4}$$

Since $\varphi\in C_0^\infty(\Omega_\varepsilon)$, it follows from the choice of ε that the maps

$$(t,x)\mapsto\varphi(x+t\omega)\quad\text{and}\quad(t,x)\mapsto\omega\cdot\nabla\varphi(x+t\omega)$$

belong to $H_0^1(0,T;L^2(\Omega))$. Hence, we have from Theorem 2.3 that

$$\alpha_\lambda\leqslant\frac{C}{\lambda}\|q_1-q_2\|_\infty,\qquad\beta_\lambda\leqslant\frac{C}{\lambda}\|q_1-q_2\|_\infty\quad\text{and}\quad\gamma_\lambda\leqslant\frac{C}{\lambda}\|q_1-q_2\|_\infty,$$

with C depending on $\|\phi\|_{H^2(0,T;L^2(\Omega))}$. Therefore,

$$\left|\int_0^T\int_\Omega\bigl(q_1(x)-q_2(x)\bigr)\bigl(\varphi(x+t\omega)\bigr)^2\,dx\,dt\right|\leqslant\frac{C}{\lambda}\|q_1-q_2\|_\infty+\delta_\lambda.$$

As the above inequality is independent of ω and we have $(\operatorname{supp}\varphi\pm t\omega)\cap\bar\Omega=\emptyset$ for all t such that $|t|>T$, we can use the Fubini theorem to obtain

$$\int_{\mathbb{R}^N}P_\omega[q_1-q_2](x)\varphi^2(x)\,dx\leqslant\frac{2C}{\lambda}\|q_1-q_2\|_\infty+2\delta_\lambda.$$

As q_1, $q_2 \in \mathcal{X}$, we have from Lemma 3.1 that there exist $\omega_1, \ldots, \omega_k \in S^{N-1}$ and $\varphi_1, \ldots, \varphi_k \in C_0^\infty(\Omega_\varepsilon)$ such that the matrix $\mathcal{A}$, with entries a_{ij} defined by (3.1), is invertible. Using those ω_j and ϕ_j, we obtain

$$\|q_1 - q_2\|_\infty \leqslant \frac{C}{\lambda}\|q_1 - q_2\|_\infty + C\delta_\lambda,$$

where δ_λ depends on T, q_1, q_2, ϕ_j and ω_j. We now choose λ such that $\frac{C}{\lambda} < 1$.

If q_1 and q_2 are such that $\Lambda_{q_1}(f_\lambda) = \Lambda_{q_2}(f_\lambda)$ for all ω_j and ϕ_j, $j = 1, 2, \ldots, k$, it follows that δ_λ is null. This implies that $q_1 = q_2$ and the proof is complete. □

Remark. Using the same arguments, we have the analogous results for the conservative wave equation. More precisely, Theorems 1.1 and 1.2 hold for the Cauchy problem

$$\begin{cases} \partial_t^2 u - \Delta u + qu = 0, & (t,x) \in Q, \\ u(0,x) = \partial_t u(0,x) = 0, & x \in \Omega, \\ u(t,\sigma) = f(t,\sigma), & (t,\sigma) \in \Sigma. \end{cases}$$

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